

Non-commutative Henselian Rings

Masood Aryapoor

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Abstract

Non-commutative Henselian rings are defined and it is shown that a local ring which is complete and separated in the topology defined by its maximal ideal is Henselian provided that it is almost commutative.

We define non-commutative Henselian rings and give some examples of them. Here, all rings are assumed to be unitary. Let us start with a definition,

Definition 1. A (possibly non-commutative) ring A is called local if all the non-invertible elements form an (two-sided) ideal which we denote by m .

If A is a local ring, then $k = A/m$ is a skew field, called the residue field. We denote the reduction map $A \rightarrow k$ by $(a \rightarrow \bar{a})$. For a brief introduction to local rings consult [Lam], Chapter 7. Let $A[x]$ be the ring of polynomials over A where the indeterminate x commutes with elements of A . Commutative Henselian rings are defined as follows,

Definition 2. Let A be a commutative local ring with the maximal ideal m and residue field k . A is called Henselian if for every polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in A[x]$ such that $\overline{f(x)} = f_1(x)f_2(x)$ for some relatively prime monic polynomials $f_i(x) \in k[x]$ then there are unique monic polynomials $F_i(x) \in A[x]$ such that $f(x) = F_1(x)F_2(x)$ and $\overline{F_i(x)} = f_i(x)$.

See [Ray] for a detailed discussion of commutative Henselian rings. The above definition makes sense as long as k , the residue field, is commutative. Therefore we have the following definition,

Definition 3. Let A be a (possibly non-commutative) local ring with the maximal ideal m and residue field k . Moreover assume that k is commutative. Then A is called Henselian if for every polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in A[x]$ such that $\overline{f(x)} = f_1(x)f_2(x)$ for some relatively prime monic polynomials $f_i(x) \in k[x]$ then there are unique monic polynomials $F_i(x) \in A[x]$ such that $f(x) = F_1(x)F_2(x)$ and $\overline{F_i(x)} = f_i(x)$.

It is well-known that every commutative local ring A which is complete and separated in the m -adic topology is Henselian. This is not true for non-commutative local rings which are complete and separated in the topology defined by the maximal ideal. However, it holds if the local ring has an extra property which we explain in what follows.

To each local ring one can associate an associative ring as follows,

Definition 4. Let A be a local ring with the maximal ideal m . Then $gr(A) = \frac{A}{m} \oplus \frac{m}{m^2} \oplus \cdots$ is defined to be the graded associated ring coming from the filtration $\cdots \subset m^{n+1} \subset m^n \subset \cdots \subset m \subset A$. A is called almost commutative if $gr(A)$ is commutative.

For basic facts regarding $gr(A)$ see [Lang].

Clearly if A is almost commutative, then k is commutative. The main theorem is,

Theorem 5. *Let A be an almost commutative local ring such that A is both separated, i.e. $\bigcap m^n = \{0\}$, and complete in the m -adic topology. Then A is a Henselian ring.*

Proof. Basically, the same proof of Hensel's lemma works. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in A[x]$ such that $\overline{f(x)} = f_1(x)f_2(x)$ for some relatively prime monic polynomials $f_i(x) \in k[x]$. We will inductively construct a sequence of monic polynomials $\{F_{1,r}(x)\}$ and $\{F_{2,r}(x)\}$ in $A[x]$ such that $\overline{F_{1,r}(x)} = f_1(x)$, $\overline{F_{2,r}(x)} = f_2(x)$, $F_{1,r+1}(x) - F_{1,r}(x) \in m^r[x]$, $F_{2,r+1}(x) - F_{2,r}(x) \in m^r[x]$ and $f(x) - F_{1,r}(x)F_{2,r}(x) \in m^r[x]$. Clearly this proves the existence part.

It is easy to find $F_{1,1}(x)$ and $F_{2,1}(x)$. Having defined $F_{1,r}(x)$ and $F_{2,r}(x)$, we define $F_{1,r+1}(x)$ and $F_{2,r+1}(x)$ as follows. Writing $F_{1,r+1}(x) = F_{1,r}(x) + G_1(x)$ and $F_{2,r+1}(x) = F_{2,r}(x) + G_2(x)$, finding $F_{1,r+1}$ and $F_{2,r+1}$ is equivalent to finding $G_1(x)$ and $G_2(x)$ in $m^r[x]$ such that $\deg(G_1(x)) < \deg(f_1(x))$, $\deg(G_2(x)) < \deg(f_2(x))$ and

$$f(x) - F_{1,r}(x)F_{2,r}(x) - G_1(x)F_{2,r}(x) - F_{1,r}(x)G_2(x) \in m^{r+1}[x].$$

By abuse of notations this is the same as finding $G_1(x)$ and $G_2(x)$ in $m^r[x]$ such that $\deg(G_1(x)) < \deg(f_1(x))$, $\deg(G_2(x)) < \deg(f_2(x))$ and $f(x) - F_{1,r}(x)F_{2,r}(x) - G_1(x)F_{2,r}(x) - F_{1,r}(x)G_2(x) = 0$ in $m^r/m^{r+1}[x]$. Considering m^r/m^{r+1} as a vector space over $k = A/m$ and using the fact that A is almost commutative, one can see that this is the same as finding $G_1(x)$ and $G_2(x)$ in $m^r[x]$ such that $\deg(G_1(x)) < \deg(f_1(x))$, $\deg(G_2(x)) < \deg(f_2(x))$ and $(f(x) - F_{1,r}(x)F_{2,r}(x)) - f_2(x)G_1(x) - f_1(x)G_2(x) = 0$ in $m^r/m^{r+1}[x]$. This is possible because $f_1(x)$ and $f_2(x)$ are relatively prime.

The uniqueness part follows from the facts that $f_1(x)$ and $f_2(x)$ are relatively prime and A is separated in the m -adic topology. \square

In the commutative case, one can use Hensel's lemma to find roots of polynomials. Next we show this connection in the non-commutative case. Let $A[x]$ be the ring of polynomials over A where the indeterminate x commutes with elements of A . So every element of $f(x) \in A[x]$ can be written uniquely as $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_i \in A$. One can consider $f(x)$ as a function on A as follows, $f(a) := a_n a^n + \cdots + a_1 a + a_0$ for $a \in A$.

Definition 6. An element $a \in A$ is called a (right) root of $f(x) = a_n x^n + \cdots + a_1 x + a_0$ if $f(a) = 0$.

We have the following proposition,

Proposition 7. An element $a \in A$ is a root of $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in A[x]$ if and only if $f(x) = g(x)(x - a)$ for some $g(x) \in A[x]$.

To see the proof and basic facts regarding right and left roots, see [Lam], Chapter 5.

Theorem 5 together with the above proposition imply that,

Theorem 8. Let A be a Henselian ring. Suppose that $\underline{f(x)} = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in A[x]$ is a monic polynomial such that $\underline{f(x)}$ has a simple root $r \in k$. Then $\underline{f(x)}$ has a unique root $a \in A$ such that $\bar{a} = r$.

In the commutative case, a local ring A is Henselian if and only if every finite A -algebra is isomorphic to a product of local rings (See [Ray]). In the non-commutative case we can give a similar criterion for Henselian rings in terms of some conditions on some modules over A .

We begin with a few definitions.

Definition 9. Let A be a ring and M a (left) A -module. We say that M is local if it has a unique maximal submodule. M is called semi-local if $M = M_1 \oplus \cdots \oplus M_k$ where M_i 's are local. It is called indecomposable if it cannot be written as $M = M_1 \oplus M_2$, where M_i 's are nonzero submodule of M . It is called strongly indecomposable if $\text{End}_A(M)$ is a local ring.

One has the following theorem.

Theorem 10. (*Krull-Schmidt-Azumaya*)

Suppose that the A -module M has the following decompositions into submodules,

$$M = M_1 \oplus \cdots \oplus M_r \simeq N_1 \oplus \cdots \oplus N_s,$$

where M_i 's are indecomposable and N_i 's are strongly indecomposable. Then $r = s$ and after a reindexing we have $M_i \simeq N_i$.

For a proof see [Lam], chapter seven.

From now on, we suppose that A is a local ring as before. Let M be an A -module. Set $\bar{M} = \frac{M}{mM}$ which is a k -module. We need a few lemmas.

Lemma 11. *Let M be an $A[x]$ -module which is a finitely generated A -module. Then, M is a local $A[x]$ -module if and only if $\frac{M}{mM}$ is a local $k[x]$ -module.*

Proof. By Nakayama's lemma, every maximal submodule of M contains mM . □

Lemma 12. *Let M, N be finitely generated A -modules. Let $\alpha : M \rightarrow N$ be an A -module homomorphism and $\bar{\alpha} : \bar{M} \rightarrow \bar{N}$ be the induced k -linear map. If $\ker(\bar{\alpha}) \neq 0$ and $\bar{\alpha}$ is onto, then $\ker(\alpha) \neq 0$.*

Proof. Suppose that v_1, \dots, v_n are elements of M such that $\bar{\alpha}(\bar{v}_1), \dots, \bar{\alpha}(\bar{v}_n)$ form a basis for \bar{N} over k . Then by Nakayama's lemma we have that $\alpha(v_1), \dots, \alpha(v_n)$ generate N as an A -module. Since $\ker(\bar{\alpha}) \neq 0$, $\bar{v}_1, \dots, \bar{v}_n$ do not generate \bar{M} . So v_1, \dots, v_n do not generate M which follows that $\ker(\alpha) \neq 0$. □

We also need the following lemma,

Lemma 13. *Let A be a local ring whose residue field k is commutative. Suppose that $p, q \in A[x]$ are polynomials of degrees r, s respectively and p is monic. If $A[x]p + A[x]q = A[x]$, then there are polynomials $p_1, q_1 \in A[x]$ such that $\deg(p_1) = \deg(q)$, $\deg(p) = \deg(q_1)$, $p_1p = q_1q$ and q_1 is monic.*

Proof. Let $\alpha : A^{s+1} \oplus A^{r+1} \rightarrow A^{r+s+1}$ be the following map,

$$\alpha(a_0, a_1, \dots, a_s, b_0, b_1, \dots, b_r) = \left(\sum_{i=0}^s a_i x^i \right) p - \left(\sum_{i=0}^r b_i x^i \right) q.$$

Using lemma 12, we have that $\ker(\alpha) \neq 0$. This shows that there are nonzero polynomials $p_1, q_1 \in A[x]$ such that $\deg(p_1) \leq \deg(q)$, $\deg(q_1) \leq \deg(p)$, $p_1 p = q_1 q$. Since \bar{p} and \bar{q} are prime in $k[x]$ and p is monic, we must have $\deg(\bar{q}_1) = \deg(p)$, hence $\deg(q_1) = \deg(p)$ and $\deg(p_1) = \deg(q)$. Finally, it is clear that q_1 can be chosen to be monic. \square

Remark 14. If $p, q \in A[x]$ are polynomials such that p is monic and $A[x]p + A[x]q + m[x] = A[x]$ then $A[x]p + A[x]q = A[x]$. In fact we have that $M = \frac{A[x]}{A[x]p + A[x]q}$ is a finitely generated A -module and $mM = M$. So, by Nakayama's lemma, $M = 0$.

We have the following theorem,

Theorem 15. *Suppose that A is a local ring whose residue field k is commutative. Then the following are equivalent,*

- (1) *A is Henselian.*
- (2) *For any monic polynomial $p \in A[x]$ the $A[x]$ -module $M = \frac{A[x]}{A[x]p}$ is semi-local.*

Proof. First we show that (1) implies (2). If \bar{p} is a power of an irreducible polynomial in $k[x]$ then $\bar{M} = \frac{M}{mM} = \frac{k[x]}{(\bar{p})}$ is a local $k[x]$ -module and by lemma 11, M is local. Suppose $\bar{p} = f_1 f_2$ where f_1 and f_2 are relatively prime polynomials of $k[x]$. By (1) we have $p = p_1 p_2 = q_2 q_1$ where p_i, q_i are monic polynomials in $A[x]$ such that $\bar{p}_i = \bar{q}_i = f_i$. This implies that $M \simeq \frac{A[x]}{A[x]p_2} \oplus \frac{A[x]}{A[x]q_1}$ because $A[x]p_2 + A[x]q_1 = A[x]$ (above remark) and it is easy to see that $A[x]p_2 \cap A[x]q_1 = A[x]p$. Now we can use induction on $\deg(p)$.

Conversely, let $p \in A[x]$ be a monic polynomial. Then we have $M = \frac{A[x]}{A[x]p} = M_1 \oplus \dots \oplus M_r$ where M_i 's are local. So we have $\bar{M} = \bar{M}_1 \oplus \dots \oplus \bar{M}_r$. On the other hand, if $\bar{p} = f_1 \dots f_s$ where f_i 's are powers of irreducible monic polynomials in $k[x]$, then $\bar{M} \simeq \frac{k[x]}{(f_1)} \oplus \dots \oplus \frac{k[x]}{(f_s)}$. It is easy to see that $\frac{k[x]}{(f_i)}$'s are strongly indecomposable as $k[x]$ -modules and \bar{M}_i 's are local, in particular

indecomposable. So by Krull-Schmidt-Azumaya theorem $r = s$ and $\bar{M}_i \simeq \frac{k[x]}{(f_i)}$ possibly after a reindexing M_i 's. If $v_i \in M_i$ is the image of $1 \in A[x]$ then $(Av_i + Axv_i + \cdots + Ax^{n_i-1}v_i) + mM_i = M_i$ where n_i is the degree of f_i . By Nakayama's lemma $(Av_i + Axv_i + \cdots + Ax^{n_i-1}v_i) = M_i$. Also $p_i v_i = 0$ for some monic polynomial p_i of degree n_i such that $\bar{p}_i = f_i$. By lemma 13, there is a monic polynomial $p' = q_1 q_2 \cdots q_r$ where q_i 's are monic polynomials and $\bar{q}_i = f_i$ and $p' \in A[x]p_i$ for each i . This implies that $p' \in A[x]p$. Since $\deg(p) = \deg(p')$ and they are monic we have $p' = p$. \square

Finally we give some examples.

Example 16. Let k be a field with a derivation. The ring of Volterra operators $k[[\partial^{-1}]]$ is defined as follows (See [Lebedev] for more on Volterra operators). It is the set of formal series $a_0 + a_1 \partial^{-1} + \cdots$ with $a_i \in k$ where $\partial^n a = \sum_{i=0}^{\infty} \binom{n}{i} a^{(i)} \partial^{n-i}$ for $n < 0$. One can see that $k[[\partial^{-1}]]$ is a local ring with the maximal ideal $m = k[[\partial^{-1}]]\partial^{-1}$ which is both separated and complete in the m -adic topology. Moreover $\text{gr}(k[[\partial^{-1}]])$ is isomorphic to $k[x]$ the ring of polynomials over k , hence commutative. So $k[[\partial^{-1}]]$ is a Henselian ring.

Example 17. If A is not almost commutative but complete and separated in the m -adic topology then there might not be any lifting of simple roots. Here is one example. Let k be a field and σ an automorphism of k . Let A be the set of all series of the form $a_0 + a_1 \tau + a_2 \tau^2 + \cdots$ where $a_i \in k$. One can make A into a ring using the relation $\tau a = \sigma(a) \tau$ for $a \in k$. Then A is a local ring which is both separated and complete in the m -adic topology and $A/m = k$ is commutative. However if σ is not the identity map then $\text{gr}(A)$ is isomorphic to the skew polynomial ring $k[x; \sigma]$, hence not commutative. Suppose $k = \mathbb{C}$ and σ is the complex conjugation. Consider the polynomial $f(x) = x^2 + 1 + \tau$ in $A[x]$. Then $\overline{f(x)}$ has a simple root in k , namely $\sqrt{-1}$. However $f(x)$ does not have any root in A . Since if $a = a_0 + a_1 \tau + a_2 \tau^2 + \cdots$ is a root of $f(x)$ then we have $0 = a^2 + 1 + \tau = a_0^2 + 1 + (a_0 a_1 + \overline{a_0} a_1 + 1) \tau + \cdots$. This implies that $a_0 = \sqrt{-1}$ or $a_0 = -\sqrt{-1}$. Therefor $a_0 a_1 + \overline{a_0} a_1 + 1 = 1 \neq 0$, a contradiction.

In the commutative case, for any local Noetherian ring A , there is a (unique) Henselian ring A^h , called the Henselization of A , and a local homomorphism $i : A \rightarrow A^h$ with the following universal property, given any local homomorphism f from A to some Henselian ring B there is a unique local

homomorphism $f^h : A^h \rightarrow B$ such that $f = f^h i$.

One can ask the same question in the non-commutative case. If A is a local ring such that $gr(A)$ is commutative, then the completion of A with respect to the m -adic topology is Henselian provided that it is separated. It is easy to see that the intersection of all local Henselian rings H in the completion \hat{A} , with the maximal ideal m_H such that $A \subset H \subset \hat{A}$ and $m_{\hat{A}} \cap H = m_H$, denoted by \bar{A} , is a Henselian local ring. In the commutative case it is not hard to see that \bar{A} is the Henselization. Therefore one might propose the following conjecture,

Conjecture 18. The Henselization exists for any almost commutative separated local ring A and $A^h \simeq \bar{A}$.

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Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven CT 06520 USA, email: masood.aryapoor@yale.edu